DOWNLOAD PDF 16. FURTHER READING ON GEOMETRIC ALGEBRA.

Chapter 1 : Geometric Algebra for Computer Science [Book]

In this video, we set up the geometric algebra of three dimensions, called G(3). We will see that the introduction of a third basis vector adds two extra bivectors and a trivector to our basis for.

Although the Clifford algebra can be defined abstractly in a coordinate-independent way, its particular realization as a specific algebra of matrices depends on which orthogonal axes the gamma matrices represent. So what precisely constitutes a "column vector" or spinor also depends on such arbitrary choices. This may or may not decompose into irreducible representations. The space of spinors may also be defined as a spin representation of the orthogonal Lie algebra. These spin representations are also characterized as the finite-dimensional projective representations of the special orthogonal group that do not factor through linear representations. Equivalently, a spinor is an element of a finite-dimensional group representation of the spin group on which the center acts non-trivially. Overview[edit] There are essentially two frameworks for viewing the notion of a spinor. One is representation theoretic. In this point of view, one knows beforehand that there are some representations of the Lie algebra of the orthogonal group that cannot be formed by the usual tensor constructions. These missing representations are then labeled the spin representations , and their constituents spinors. These double covers are Lie groups , called the spin groups Spin n or Spin p, q. All the properties of spinors, and their applications and derived objects, are manifested first in the spin group. Representations of the double covers of these groups yield double-valued projective representations of the groups themselves. This means that the action of a particular rotation on vectors the quantum Hilbert space is only defined up to a sign. The other point of view is geometrical. One can explicitly construct the spinors, and then examine how they behave under the action of the relevant Lie groups. This latter approach has the advantage of providing a concrete and elementary description of what a spinor is. However, such a description becomes unwieldy when complicated properties of spinors, such as Fierz identities , are needed. Clifford algebra The language of Clifford algebras [3] sometimes called geometric algebras provides a complete picture of the spin representations of all the spin groups, and the various relationships between those representations, via the classification of Clifford algebras. It largely removes the need for ad hoc constructions. In detail, let V be a finite-dimensional complex vector space with nondegenerate bilinear form g. It is an abstract version of the algebra generated by the gamma or Pauli matrices. If n is odd, this Lie algebra representation is irreducible. Irreducible representations over the reals in the case when V is a real vector space are much more intricate, and the reader is referred to the Clifford algebra article for more details. Spinors form a vector space , usually over the complex numbers , equipped with a linear group representation of the spin group that does not factor through a representation of the group of rotations see diagram. The spin group is the group of rotations keeping track of the homotopy class. Spinors are needed to encode basic information about the topology of the group of rotations because that group is not simply connected , but the simply connected spin group is its double cover. So for every rotation there are two elements of the spin group that represent it. Geometric vectors and other tensors cannot feel the difference between these two elements, but they produce opposite signs when they affect any spinor under the representation. Thinking of the elements of the spin group as homotopy classes of one-parameter families of rotations, each rotation is represented by two distinct homotopy classes of paths to the identity. If a one-parameter family of rotations is visualized as a ribbon in space, with the arc length parameter of that ribbon being the parameter its tangent, normal, binormal frame actually gives the rotation , then these two distinct homotopy classes are visualized in the two states of the belt trick puzzle above. The space of spinors is an auxiliary vector space that can be constructed explicitly in coordinates, but ultimately only exists up to isomorphism in that there is no "natural" construction of them that does not rely on arbitrary choices such as coordinate systems. A notion of spinors can be associated, as such an auxiliary mathematical object, with any vector space equipped with a quadratic form such as Euclidean space with its standard dot product , or Minkowski space with its Lorentz metric. In the latter case, the "rotations" include the Lorentz boosts , but otherwise the theory is substantially similar. See Special unitary group. In even dimensions, this representation is reducible when taken as a representation of

Spin p, q and may be decomposed into two: Dirac and Weyl spinors are complex representations while Majorana spinors are real representations. The Dirac, Lorentz, Weyl, and Majorana spinors are interrelated, and their relation can be elucidated on the basis of real geometric algebra. The classical neutrino of the standard model of particle physics is an example of a Weyl spinor. However, because of observed neutrino oscillation , it is now believed that they are not Weyl spinors, but perhaps instead Majorana spinors. In , an international team led by Princeton University scientists announced that they had found a quasiparticle that behaves as a Weyl fermion. Spin representation One major mathematical application of the construction of spinors is to make possible the explicit construction of linear representations of the Lie algebras of the special orthogonal groups , and consequently spinor representations of the groups themselves. At a more profound level, spinors have been found to be at the heart of approaches to the Atiyahâ€”Singer index theorem , and to provide constructions in particular for discrete series representations of semisimple groups. Whereas the weights of the tensor representations are integer linear combinations of the roots of the Lie algebra, those of the spin representations are half-integer linear combinations thereof. Explicit details can be found in the spin representation article. Attempts at intuitive understanding[edit] The spinor can be described, in simple terms, as "vectors of a space the transformations of which are related in a particular way to rotations in physical space".

Chapter 2 : Geometric algebra - Wikipedia

Understanding geometric algebra for electromagnetic theory / "This book aims to disseminate geometric algebra as a straightforward mathematical tool set for working with and understanding classical electromagnetic theory.

We wrote this book to explain the basic structure of geometric algebra, and to help the reader become a practical user. We employ various tools to get there: When reading it, you should remember that geometric algebra is fundamentally simple, and fundamentally simplifying. That simplicity will not always be clear; precisely because it is so fundamental, it does basic things in a slightly different way and in a different notation. This requires your full attention, notably in the beginning, when we only seem to go over familiar things in a perhaps irritatingly different manner. The patterns we uncover, and the coordinate-free way in which we encode them, will all pay off in the end in generally applicable quantitative geometrical operators and constructions. The book should also be well suited for selfstudy at the post-graduate level; in fact, we tried to write the book that we would have wanted ourselves for this purpose. Depending on your level of interest, you may want to read it in different ways. In a comfortable reading, you can absorb what is different in geometric algebra, and its structure will help you understand all those old tricks in your library. In our experience, it makes many arcane techniques comprehensible, and it helped us to learn from useful math books that we would otherwise never have dared to read. And who knowsâ€"you may come back for more. The advantages for the previous category will apply to you as well. This geometric algebra way of thinking is quite natural, and we are rather envious that you can learn it from scratch, without having to unlearn old methods. Our style in this book is factual. We give you the necessary mathematics, but always relate the algebra to the geometry, so that you get the complete picture. Occasionally, there is a need for more extensive proofs to convince you of the consistency of aspects of the framework. When such a proof became too lengthy and did not further the arguments, it was relegated to an appendix. The derivations that remain in the text should be worth your time, since they are good practice in developing your skills. By its very nature, geometric algebra collates many partial results in a single framework, and the original sources become hard to trace in their original context. It is part of the pleasure of geometric algebra that it empowers the user; by mastering just a few techniques, you can usually easily rediscover the result you need. Once you grasp its essence, geometric algebra will become so natural that you will wonder why we have not done geometry this way all along. The reason is a history of geometric mis representation, for almost all elements of geometric algebra are not newâ€"in hindsight. This gave us the mixed blessing of coordinates, and the tiresome custom of specifying geometry at the coordinate level whereas coordinates should be relegated to the lowest implementational level, reserved for the actual computations. To have a more direct means of expression, Hermann Grassmann â€" developed a theory of extended quantities, allowing geometry to be based on more than points and vectors. Unfortunately, his ideas were ahead of their time, and his very compact notation made his work more obscure than it should have been. All these individual contributions pointed toward a geometric algebra, and at the end of the 19th century, there were various potentially useful systems to represent aspects of geometry. Gibbs â€" made a special selection of useful techniques for the 3D geometry of engineering, and this limited framework is basically what we have been using ever since in the geometrical applications of linear algebra. Linear algebra and matrices, with their coordinate representations, became the mainstay of doing geometry, both in practice and in mathematical development. He rescued the half-forgotten geometric algebra by now called Clifford algebra and developed in nongeometric directions , developed it into an alternative to the classical linear algebraâ€"based representations, and started advocating its universal use. In the s, his voice was heard, and with the implementation of geometric algebra into interactive computer programs its practical applicability is becoming more apparent. Gibbs was wrong in assuming that computing with the geometry of 3D space requires only representations of 3D points, although he did give us a powerful system to compute with those. This book will demonstrate that allowing more extended quantities in higher-dimensional representational spaces provides a more convenient executable language for geometry. Maybe we could have had this all along; but perhaps we indeed needed to wait for the arrival of computers to appreciate the effectiveness of this approach. All three were developed by us, and can be found on the web site: It was originally developed as a teaching tool, and a web tutorial is available, using GAViewer to explain the basics of geometric algebra. You can use GAViewer when reading the book to type in algebraic formulas and have them act on geometrical elements interactively. This interaction should aid your understanding of the correspondence between geometry and algebra considerably. The GA sandbox source code package used for the programming examples and exercises in this book is built on top of Gaigen 2. To compile and run the programming examples in Part I and Part II, you only have to download the sandbox package from the web site. It is written in Java and intended to show only the essential structure; we do not deem it usable for anything that is computationally intensive, since it can easily be 10 to times slower than Gaigen 2. PREFACE xxxv If you are serious about implementing further applications, you can start with the GA sandbox package, or other available implementations of geometric algebra, or even write your own package. We are grateful to Joan Lasenby of Cambridge University for her detailed comments on the early chapters, and for providing some of the applied examples. We are also indebted to Timaeus Bouma for his keen insights that allowed our software to be well-founded in mathematical fact. Ultimately, though, this book would have been impossible without the home front: Daniel Fontijne owes many thanks to Yvonne for providing the fun and artistic reasons to study geometric algebra, and to Femke and Tijmen for the many refreshing breaks while working at home. Stephen Mann would like to thank Mei and Lilly for their support during the writing of this book. This book is about geometric algebra, a powerful computational system to describe and solve geometrical problems. You will see that it covers familiar groundâ€”lines, planes, spheres, rotations, linear transformations, and moreâ€”but in an unfamiliar way. Our intention is to show you how basic operations on basic geometrical objects can be done differently, and better, using this new framework. It helps us discuss some of the important properties of the computational framework. You should of course read between the lines: This is depicted in Figure 1. Here is how geometric algebra encodes this in its conformal model of Euclidean geometry: The three points are denoted by three elements $c1$, $c2$, and $c3$. The outer product is antisymmetric:

Chapter 3 : Spinor - Wikipedia

Further Reading Null Geometric Algebra Approach A point in the Euclidean plane is represented by anull vector 16/ Motivation Projective Incidence Geometry.

Overview Contemporary geometry has many subfields: Differential geometry uses techniques of calculus and linear algebra to study problems in geometry. It has applications in physics , including in general relativity. Topology is the field concerned with the properties of geometric objects that are unchanged by continuous mappings. In practice, this often means dealing with large-scale properties of spaces, such as connectedness and compactness. Convex geometry investigates convex shapes in the Euclidean space and its more abstract analogues, often using techniques of real analysis. It has close connections to convex analysis , optimization and functional analysis and important applications in number theory. Algebraic geometry studies geometry through the use of multivariate polynomials and other algebraic techniques. It has applications in many areas, including cryptography and string theory. Discrete geometry is concerned mainly with questions of relative position of simple geometric objects, such as points, lines and circles. It shares many methods and principles with combinatorics. Computational geometry deals with algorithms and their implementations for manipulating geometrical objects. Although being a young area of geometry, it has many applications in computer vision , image processing , computer-aided design , medical imaging , etc. History of geometry A European and an Arab practicing geometry in the 15th century. The earliest recorded beginnings of geometry can be traced to ancient Mesopotamia and Egypt in the 2nd millennium BC. For example, the Moscow Papyrus gives a formula for calculating the volume of a truncated pyramid, or frustum. These geometric procedures anticipated the Oxford Calculators , including the mean speed theorem , by 14 centuries. Around BC, geometry was revolutionized by Euclid, whose Elements , widely considered the most successful and influential textbook of all time, [18] introduced mathematical rigor through the axiomatic method and is the earliest example of the format still used in mathematics today, that of definition, axiom, theorem, and proof. Although most of the contents of the Elements were already known, Euclid arranged them into a single, coherent logical framework. The Satapatha Brahmana 3rd century BC contains rules for ritual geometric constructions that are similar to the Sulba Sutras. They contain lists of Pythagorean triples , [22] which are particular cases of Diophantine equations. The Bakhshali manuscript also "employs a decimal place value system with a dot for zero. Chapter 12, containing 66 Sanskrit verses, was divided into two sections: This was a necessary precursor to the development of calculus and a precise quantitative science of physics. The second geometric development of this period was the systematic study of projective geometry by Girard Desargues â€" Projective geometry is a geometry without measurement or parallel lines, just the study of how points are related to each other. Two developments in geometry in the 19th century changed the way it had been studied previously. As a consequence of these major changes in the conception of geometry, the concept of "space" became something rich and varied, and the natural background for theories as different as complex analysis and classical mechanics. Important concepts in geometry The following are some of the most important concepts in geometry. Euclidean geometry Euclid took an abstract approach to geometry in his Elements , one of the most influential books ever written. Euclid introduced certain axioms , or postulates , expressing primary or self-evident properties of points, lines, and planes. He proceeded to rigorously deduce other properties by mathematical reasoning. Point geometry Points are considered fundamental objects in Euclidean geometry. However, there has been some study of geometry without reference to points. Line geometry Euclid described a line as "breadthless length" which "lies equally with respect to the points on itself". For instance, in analytic geometry , a line in the plane is often defined as the set of points whose coordinates satisfy a given linear equation , [34] but in a more abstract setting, such as incidence geometry , a line may be an independent object, distinct from the set of points which lie on it. Plane geometry A plane is a flat, two-dimensional surface that extends infinitely far. For instance, planes can be studied as a topological surface without reference to distances or angles; [37] it can be studied as an affine space , where collinearity and ratios can be studied but not distances; [38] it can be studied as the complex plane using techniques of complex analysis ; [39] and so on. Angle Euclid defines a plane angle as the inclination to each other, in a plane, of two lines which meet each other, and do not lie straight with respect to each other. The acute and obtuse angles are also known as oblique angles. In Euclidean geometry , angles are used to study polygons and triangles , as well as forming an object of study in their own right. Curve geometry A curve is a 1-dimensional object that may be straight like a line or not; curves in 2-dimensional space are called plane curves and those in 3-dimensional space are called space curves. A surface is a two-dimensional object, such as a sphere or paraboloid. In algebraic geometry, surfaces are described by polynomial equations. Manifold A manifold is a generalization of the concepts of curve and surface. In topology , a manifold is a topological space where every point has a neighborhood that is homeomorphic to Euclidean space. The Pythagorean theorem is a consequence of the Euclidean metric. A topology is a mathematical structure on a set that tells how elements of the set relate spatially to each other. Other important examples of metrics include the Lorentz metric of special relativity and the semi- Riemannian metrics of general relativity. Compass and straightedge constructions Classical geometers paid special attention to constructing geometric objects that had been described in some other way. Classically, the only instruments allowed in geometric constructions are the compass and straightedge. Also, every construction had to be complete in a finite number of steps. However, some problems turned out to be difficult or impossible to solve by these means alone, and ingenious constructions using parabolas and other curves, as well as mechanical devices, were found. The concept of dimension has gone through stages of being any natural number n, to being possibly infinite with the introduction of Hilbert space , to being any positive real number in fractal geometry. Dimension theory is a technical area, initially within general topology , that discusses definitions; in common with most mathematical ideas, dimension is now defined rather than an intuition. Connected topological manifolds have a well-defined dimension; this is a theorem invariance of domain rather than anything a priori. The issue of dimension still matters to geometry as many classic questions still lack complete answers. For instance, many open problems in topology depend on the dimension of an object for the result. In physics, dimensions 3 of space and 4 of space-time are special cases in geometric topology , and dimensions 10 and 11 are key ideas in string theory. Currently, the existance of the theoretical dimensions is purely defined by technical reasons; it is likely that further research may result in a geometric reason for the significance of 10 or 11 dimensions in the theory, lending credibility or possibly disproving string theory. Symmetry A tiling of the hyperbolic plane The theme of symmetry in geometry is nearly as old as the science of geometry itself. Symmetric shapes such as the circle , regular polygons and platonic solids held deep significance for many ancient philosophers and were investigated in detail before the time of Euclid. Symmetric patterns occur in nature and were artistically rendered in a multitude of forms, including the graphics of M. Nonetheless, it was not until the second half of 19th century that the unifying role of symmetry in foundations of geometry was recognized. Symmetry in classical Euclidean geometry is represented by congruences and rigid motions, whereas in projective geometry an analogous role is played by collineations , geometric transformations that take straight lines into straight lines. Both discrete and continuous symmetries play prominent roles in geometry, the former in topology and geometric group theory , the latter in Lie theory and Riemannian geometry. A different type of symmetry is the principle of duality in projective geometry see Duality projective geometry among other fields. This meta-phenomenon can roughly be described as follows: A similar and closely related form of duality exists between a vector space and its dual space. Non-Euclidean geometry Differential geometry uses tools from calculus to study problems involving curvature. In the nearly two thousand years since Euclid, while the range of geometrical questions asked and answered inevitably expanded, the basic understanding of space remained essentially the same. Immanuel Kant argued that there is only one, absolute, geometry, which is known to be true a priori by an inner faculty of mind: Euclidean geometry was synthetic a priori. They demonstrated that ordinary Euclidean space is only one possibility for development of geometry. Contemporary geometry Euclidean geometry Geometry lessons in the 20th century Euclidean geometry has become closely connected with computational geometry , computer graphics , convex geometry , incidence geometry , finite geometry , discrete geometry , and some areas of combinatorics. Attention was given to further work on Euclidean geometry and the Euclidean groups by crystallography and the work of H. Coxeter , and can be seen in theories of Coxeter groups and polytopes. Geometric group theory is an expanding area of the theory of more general discrete groups , drawing on geometric models and algebraic techniques. Contemporary differential geometry is intrinsic, meaning that the spaces it considers are smooth manifolds whose geometric structure is governed by a Riemannian metric , which determines how distances are measured near each point, and not a priori parts of some ambient flat Euclidean space. Topology and geometry A thickening of the trefoil knot The field of topology , which saw massive development in the 20th century, is in a technical sense a type of transformation geometry , in which transformations are homeomorphisms. Contemporary geometric topology and differential topology , and particular subfields such as Morse theory , would be counted by most mathematicians as part of geometry. Algebraic topology and general topology have gone their own ways. From late s through mids it had undergone major foundational development, largely due to work of Jean-Pierre Serre and Alexander Grothendieck. This led to the introduction of schemes and greater emphasis on topological methods, including various cohomology theories. One of seven Millennium Prize problems , the Hodge conjecture , is a question in algebraic geometry. The study of low-dimensional algebraic varieties, algebraic curves , algebraic surfaces and algebraic varieties of dimension 3 "algebraic threefolds" , has been far advanced. Arithmetic geometry is an active field combining algebraic geometry and number theory. Other directions of research involve moduli spaces and complex geometry. Algebro-geometric methods are commonly applied in string and brane theory. Applications Geometry has found applications in many fields, some of which are described below. Art Mathematics and art are related in a variety of ways. For instance, the theory of perspective showed that there is more to geometry than just the metric properties of figures: Mathematics and architecture and Architectural geometry Mathematics and architecture are related, since, as with other arts, architects use mathematics for several reasons. Apart from the mathematics needed when engineering buildings, architects use geometry: Physics The polytope , orthogonally projected into the E8 Lie group Coxeter plane. Lie groups have several applications in physics. The field of astronomy , especially as it relates to mapping the positions of stars and planets on the celestial sphere and describing the relationship between movements of celestial bodies, have served as an important source of geometric problems throughout history. Modern geometry has many ties to physics as is exemplified by the links between pseudo-Riemannian geometry and general relativity. One of the youngest physical theories, string theory , is also very geometric in flavour.

Chapter 4 : Alan Macdonald: Geometric Algebra and Foundations of Physics

Not only does geometric algebra provide us with new ways to reason about computational geometry, it also embeds and explains all existing theories in- cluding complex numbers, quaternions, matrix-algebra, and PlucÂ¨ kerspace.

Elsewhere on the site, you may find a detailed table of contents , examples of specific chapters , and examples of specific programming exercises. The example reveals its main characteristics for computations in Euclidean geometry: We show how a GA-based computer program looks, and explain the structure of the book. This chapter can be viewed here in its entirety. It provides the algebraic structure, with products inspired by the desire to represent linear subspaces and geometric operations on them. Throughout, the concepts are illustrated and developed through drills, structural exercises, and programming exercises. To extend the representational capabilities of linear algebra, Chapter 2: Spanning Oriented Subspaces introduces the outer product. The outer product of two vectors is algebraically a 2-blade. In its most simple geometric interpretation such a 2-blade represents the 2-dimensional subspace through the origin, spanned by the vectors, as shown in Figure 2. An outer product of three vectors is a 3-blade, see Figure 2. A 3-blade represents the volume spanned by the three vectors. Such extended geometrical entities are now basic elements of algebraic computation. We use the blades of a geometric algebra to algebraically represent all geometrical primitives. The scalars in a vector space are represented as 0-blades, the vectors by 1-blades, and the oriented area elements are 2-blades. In Part II, we will give enriched interpretations to these blades. For example, in the homogeneous model 2-blades are used to represent lines and 3-blades represent planes; in the conformal model 3-blades represent circles, and so on. Metric Products of Subspaces, we generalize the dot product from linear algebra such that it is applicable to blades. These metric products allow us to measure angles and contents, orthogonalize vectors, perform projections, and so on. It is an elementary construction, more fundamental than projection because it is linear in both x and B. In this chapter, we also introduce dualization. Any element in geometric algebra has a dual representation, through its inner product relative to the total volume element of the space. Extending a linear transformation to blades. Linear Transformations of Subspaces shows that the usual linear transformations defined on vectors 1-blades can be applied to k-blades, using the same formula. This enriches the descriptive and computational power of linear algebra considerably, since you can now apply linear transformations immediately to subspaces, without needing to decompose those into vectors first. These linear transformations preserve the properties of the outer product, and are hence called outermorphisms. This chapter is a bit algebraic, but it is important to know how the subspace products transform under linear transformations. The meet of two 2-blades. Intersections and Unions of Subspaces, we codify those geometric operations algebraically, as the meet and join products of blades. They are algebraically a bit involved, and algorithmically somewhat expensive. Yet their geometric importance warrants the effort to generalize them: This chapter is somewhat abstract, and can be skimmed at first reading. The Fundamental Product of Geometric Algebra, the magic of geometric algebra really begins to show. It is surprisingly simple algebraically - being linear, associative, and giving scalar squares for vectors defines it fully. In this chapter, we carefully relate it to the outer product and contraction of the earlier chapters, to provide some geometric intuition for its structure and use. The geometric product is invertible, and this immediately simplifies the algebraic specification of a specific geometrical situation. You can divide by vectors, and in the planar example of Figure 6. A rotation is a double reflection. This is a universal, structure-preserving representation of the action of an operator onto any element: The transform of a geometric combination of elements equals the combination of the transforms of the elements. Any framework for geometry should obey this; what is nice about geometric algebra used properly is that it has this structure preservation inherently built in: Actually, only orthogonal transformations have this desirable property; but in Part II we show techniques to ensure that important geometrical transformations become orthogonal in their representing algebra. The most basic versor is a vector v, and the associated transformation a reflection in the plane perpendicular to v. Multiple applications of such versors then generates rotation versors, a. These have all the properties of complex numbers and quaternions, but now in n-D and universally applicable in a real vector space context. As a mirror rotates slightly, its reflections move. By a first order approximation in geometric differentiation, we easily find the locally circular orbit of a caustic. Geometric Differentiation is about the process of computing with small changes in geometric quantities. When the changes are small, those computations can be linear to a good approximation, and it is not too hard to develop a calculus for geometry by analogy to classical analysis. When formulated with geometric algebra, it becomes possible to differentiate not only with respect to a scalar as in real calculus or a vector as in vector calculus , but also with respect to general multivectors and k-blades. The differentiation operators follow the rules of geometric algebra: As you might expect, this has precisely the right geometric consequences for the differentiation process to give geometrically significant results. In this book, we only give the basic principles, to allow you to interface with other texts that contain a more complete differential geometry in the GA formulation. Modeling Geometries introduces the procedure of Part II: As we do so, we are able to apply it to specific situations, permitting increasingly useful programming exercises. The Vector Space Model - The Algebra of Directions is the first geometric model, and the most direct visualization of the structure of geometric algebra. All of Part I was in fact about characterizing k-dimensional directions. We develop that correspondence in more detail for a Euclidean space. This leads naturally to applications like: In all of those applications, the geometric product immensely simplifies the expression and derivation of advanced results. Lines as 2-blades in the homogeneous model. The Homogeneous Model embeds the familiar homogeneous coordinates used throughout computer graphics, robotics and computer vision into geometric algebra. This clarifies and extends its algebraic structure. Its basic elements are offset flats in Euclidean space, and these are the blades of its geometric algebra. The outer product between two 1-blades representing geometric points is the 2-blade that represents the geometric line between those points see Figure Naturally equivalent representations of the same geometry such as by a point and a direction vector become algebraically identical elements. The homogeneous model connects well to existing software for computer graphics and computer vision, and we demonstrate that by the programming examples of this chapter. A pinhole camera with a plane of rays. Applications of the Homogeneous Model treats two obvious uses of the homogeneous model for non-metric applications: Imaging by multiple cameras in its usual treatment gets obscured by the need to represent the relevant operations in coordinate form; here, too, the homogeneous model provides algebraic insight into the straightforward essence of the geometry-based techniques. Screw representation of rigid body motions in the conformal model. It applies the modeling principles to design an algebra that is automatically structure-preserving for the constructions of n-dimensional Euclidean geometry. This conformal model is the geometric algebra of a representational space with two extra dimensions. One dimension represents the point at the origin as in the homogeneous model and the other the unique point at infinity. These are null dimensions, contributing to the metric properties in a subtle but convenient way. Differentiation of a reflected line in a rotating mirror. The defining property of the conformal model is that the inner product between 1-blades represents the squared distance between Euclidean points. This guarantees that Euclidean motion are represented by orthogonal transformations. Those are coded as versors, and that results in the structure-preservation. The logarithm of a rigid body motion can then be derived in closed form, permitting screw-like interpolation of such motions Figure The Euclidean intersection of circles computed as the meet of representing planes. New Primitives for Euclidean Geometry, we show that the blades of the conformal model can represent an algebraically consistent catalog of elements that are useful in Euclidean geometry. They give us spheres, circles, point pairs and tangents as direct elements of computation. We carefully develop the representation of these new elements and show how to retrieve their parameters. We also give the geometry behind the algebra of the conformal model, and show in detail how the universal meet of flat subspaces can perform the intersection of circles and spheres. The meet and plunge of spheres. Constructions in Euclidean Geometry. The algebraically consistent structure provides universal constructions in Euclidean geometry. Some of those are unusual, but useful: Normal vector, position vector, free vector, line vector and tangent vector now all automatically move in the correct way under the same Euclidean versors. Even with all the new techniques for Euclidean geometry of the previous three chapters, the possibilities of the conformal model are not exhausted. Conformal Operators we show that the versor of the conformal model represent conformal transformations that preserve angles Figure These include reflection in a sphere, and uniform scaling; Euclidean motions were just a special case. The non-Euclidean hyperbolic and elliptic geometries can now also be processed in the same model Figure Operational Models for Geometries sums up the principles that guided the design of structure preserving models in Part II, and makes an inventory of geometries for which we know such operational models. More will presumably be found, all obeying the same design principles - the conformal model is merely the first. The conformal model of a 3D Euclidean space uses a dimensional geometric algebra, and a naive implementation would be prohibitively slow. In this Part III, we show that you can use the structure and sparseness of geometric algebra to compete in efficiency with the usual, coordinate-based, hand-coded minimal solutions for applications such as ray-tracing. The coordinate-free specifications in the geometric algebra code then truly acts like a high-level programming language for geometry.

Chapter 5 : Maths - Clifford / Geometric Algebra - Martin Baker

The geometric algebra (GA) of a vector space is an algebra over a field, noted for its multiplication operation called the geometric product on a space of elements called multivectors, which is a superset of both the scalars and the vector space.

Geometric Algebra is as an extension of Vector Algebra When we discussed vector algebra we had two types of multiplication: A dot product takes two vectors and produces a scalar. A cross product takes two vectors and produces a bivector. Having two types of multiplication which give outputs that are not vectors is not altogether satisfactory. This is not possible for vector algebra but it is possible if our elements are a superset of scalars, vectors, bivectors and higher order components. We can then define a general multiplication which is a combination of cross and dot multiplication. As a first stage imagine a 3D vector as a linear sum of base vectors e1, e2 and e3. However the operations are still not general in that these operations only apply to a subset of the multivectors. This type of multiplication has similar properties to the vector cross product but it applies to any number of dimensions. The reason that the number of types does not keep growing is that the above rules only apply if the vectors are different, if we multiply the vector by itself the result is zero so: This type of algebra, where a quantity is not zero, but squaring it makes it zero is known as a grassmann algebra. So the quantities in this algebra consist of 2n scalar values some of which may be zero and therefore not written. In the same way that vector algebra may be different depending on the dimension of the vectors, geometric algebra depends on the dimension of the vectors contained in it. The elements equivalents to numbers - i. These multivectors are made up of blades and each blade has a grade as well as a dimension. Grade 0 is a scalar number which has dimension 1 , Grade 1 is a vector whose dimension determines the number and dimension of the remaining grades , Grade 2 is a bivector whose dimension depends on the vector , And so on for tri-vectors Hopefully this will become clearer later when we derive multivectors from different dimension vectors. Perhaps we can start here with three dimensional vector algebra, the rules are changed slightly, in a way that generalises the result. We replace the dot, cross and scalar products of vector algebra with the following operations for geometric algebra: The small change in the rules is that the outer product of two vectors results in a new entity called a bivector, there may also be tri-vectors and so on. Multiplying by a vector using the outer product increases the grade of the result by 1, multiplying by a vector using the inner product decreases the grade of the result by 1, so these operations have a nice symmetry. In the general case a multivector may contain all possible combinations not permutations of the base vectors. When we are working with vectors the geometric product is a sum of the inner and outer products: It would be good to define when this holds, obviously it depends what type of inner product we use. I would like to understand more about the rules for inner and outer products. Choice of bases If we want to have a standardised representation of multivectors then we need to choose: Possible options for factor ordering are:

DOWNLOAD PDF 16. FURTHER READING ON GEOMETRIC ALGEBRA.

Chapter 6 : Geometric algebra for computer science (with errata) - PDF Free Download

Maths - clifford / Geometric Algebra - Further Reading Clifford Algebra to Geometric Calculus: A Unified Language for Mathematics and Physics (Fundamental Theories of Physics). This book is intended for mathematicians and physicists rather than programmers, it is very theoretical.

Did you know that the inner product space R n can be embedded in a vector space of dimension 2 n which is also an associative algebra with unit, the geometric algebra? Some members of the geometric algebra represent geometric objects in R n. Other members represent geometric operations on the geometric objects. Geometric algebra and its extension to geometric calculus unify, simplify, and generalize vast areas of mathematics that involve geometric ideas, including linear algebra, multivariable calculus, real analysis, complex analysis, and euclidean, noneuclidean, and projective geometry. They provide a unified mathematical language for physics classical and quantum mechanics, electrodynamics, relativity , the geometrical aspects of computer science e. Linear and Geometric Algebra This textbook for the first undergraduate linear algebra course presents a unified treatment of linear algebra and geometric algebra, while covering a majority of the usual linear algebra topics. Fourth printing, corrected and slightly revised. There is a new chapter on the conformal model. Videos I have created a six video YouTube playlist Geometric Algebra , about 72 minutes in all, taken from the book. Unlike the book, some knowledge of linear algebra is a prerequisite for the videos. The book assumes no previous knowledge of linear algebra. This textbook for the first undergraduate vector calculus course presents a unified treatment of vector calculus and geometric calculus, while covering a majority of the usual vector calculus topics. Videos I have created a five video YouTube playlist Geometric Calculus , about 53 minutes in all, taken from the book. It is a sequel to my Geometric Algebra playlist. Unlike the book, some knowledge of vector calculus is a prerequisite for the videos. The book assumes no previous knowledge of vector calculus. Interview Professor Ahmad Eid, publisher of Geometric Algebra Explorer, asked me for my thoughts about geometric algebra and its place in undergraduate education. Therefore his calculus does not have a proper foundation. The paper is an introduction to geometric algebra and geometric calculus for those with a knowledge of undergraduate mathematics. No knowledge of physics is required. The section Further Study lists many papers available on the web.

Chapter 7 : Vector calculus identities

In this video, we will look at the algebraic and geometric interpretations of adding bivectors together in G(3). We will also show that all the bivectors in G(2) and G(3) can be factored as a.

However, Quaternions are hard to understand because they are taught at face value. We sort of just accept their odd multiplication tables and other arcane definitions and use them as black boxes that rotate vectors in the ways we want. Who cares as long as it rotates my vectors the right way, right? There is a way to represents rotations called a Rotor that subsumes both Complex Numbers in 2D and Quaternions in 3D and even generalizes to any number of dimensions. We can build Rotors almost entirely from scratch, instead of defining quaternions out of nowhere and trying to explain how they work retroactively. This takes more time, but I find it is very much worth it because it makes them much easier to understand! Also, 3D Rotors do not require the use of a fourth dimension of space in order to be visualized and understood. It would be great if we could start phasing out the use and teaching of Quaternions and replace them with Rotors. The change is simple and the code remains almost the same. Anything you can do with a Quaternion, such as Interpolation and avoiding Gimbal lock, you can do on a Rotor. But the understanding grows a lot. In the following article, every diagram is interactive. The video follows the article, and you can press the buttons to play the relevant section of video. Conversely, you can press the button to go to the section of the article that corresponds to what the video is playing at this moment. You can maximize your window to have more space for the video, or you can press the button to set it to a fixed size. Planes of Rotations Rotations happen in 2D planes In 3D, we usually think of rotations as happening around an axis, like a wheel turning around its axle, but instead of thinking about the axle a more correct way is to think about the plane that the wheel lies on, perpendicular to the axle. If you told a 2D "flatlander" who lives inside a 2D plane and has only ever experienced 2D about a perpendicular rotation axis they would look at you and ask "which direction does the axis point along? However, if we think about rotations as happening inside planes, the sense is clear: That convention becomes unnecessary when we talk directly about the plane itself. But why "leave" the plane, since a rotation is fundamentally a 2D thing? If the cross product creates the normal vector to a plane, the outer product creates the plane itself. Taking the normal to the plane is extraneous. The idea of a bivector might seem a bit strange at first, but they are pretty much as fundamental as vectors, as we will see. If a vector is like a line, then a bivector is like a plane The properties of the outer product are suited to capture the properties of planes. Basis for Bivectors Bivectors have components, just like vectors. But they are defined in terms of basis planes instead of basis lines like vectors. So a 2D bivector only has one component. You can see that by changing the angle between the vectors the area of the parallelogram changes according to the sine of the angle. This simple property defines what a bivector is: Swapping the arguments changes the sign of the result this is called "anti-symmetric". In the diagram, the sign is represented using the color, which changes from blue to green. You can see how the properties of the outer product are suited to capture the properties of planes and rotations. The signed area of the parallelogram is proportional to the lengths of both vectors: So for example doubling the length of one vector doubles the area. We can get the actual value by plugging in the vectors in component form: The projections of the vector are the lengths of that vector along each basis vector, while the projections of the bivector are the areas of the plane on each basis plane. The components of a 3D bivector are just the three 2D projections of the bivector onto the 2D basis planes. Using the same method as before we find that the actual values of the components look a lot like the XY component from the 2D case, but applied to all three planes: Does the exterior product remind you of anything? In 3D, the definition of the outer product is very similar to that of the cross product. In fact, in 3D a vector that comes from a cross product such as a normal vector will have three components which are equal to the components of the bivector the numbers are the same, but the basis is different. I remember thinking when I was learning the cross product, why the hell does it return a vector that has length equal to the area of the parallelogram formed by the two vectors? That feels so arbitrary. And why would you be allowed to turn the area of the parallelogram into the length of the vector? Semantics of Vectors and Bivectors In 3D, a bivector has three coordinates, one per plane: Each plane is perpendicular to one axis. In programming terms, they both have the same memory layout, but different operations. Using a 3D vector instead of a 3D bivector is like "type-casting" the bivector. Bivector is the actual "type" of the object and it should be thought of and manipulated as such. Just like in 2D there is only one plane which fills all of 2D space, in 3D there is only one volume which fills all of 3D space. The geometric product is defined so that vectors have inverses i. The goal is to be able to multiply vectors together so that â€”just like for matricesâ€" multiplication corresponds to geometric operations. Together they fully describe the angle between the vectors, as well the plane they form. So the geometric product is: However this is similar to how a complex number is the sum of a scalar and an "imaginary" number, so you might be used to it already. Here the bivector part corresponds to the "imaginary" part of the complex number. The geometric product also gives these "property bundles" operations that can be applied to them, and these operations have geometric interpretations for example: Multiplication Table The multiplication table helps make this product more concrete: This gives the following table:

Chapter 8 : Geometric Algebra For Computer Science

Using geometric algebra makes it easy to read off this formula and determine what is going to happen, i.e. the e 1 vector is going to be operated on via geometric product and the result will be another vector that is rotated Ï‰ t radians in a counter-clockwise direction.

Chapter 9 : Geometric Algebra - [PDF Document]

In order to represent 3D solid bodies using geometric algebra we need to use 4D multivectors as will be described on further pages. This means that we need 16 scalar variables to represent isometries, the same as a 4x4 matrix, which can also represent isometries and most people find matrices easier to understand.